DO FINANCIAL PLANNERS TAKE FINANCIAL CRASHES IN THEIR ADVICE: DYNAMIC ASSET ALLOCATION UNDER THICK TAILS AND FAST VOLATILITY UPDATING

Ahmad Telfah

API/WPS 0604

Correspondence

Dr. Ahmad Telfah, The Arab Planning Institute, P.O. Box: 5834 Safat. 13059 Kuwait,

Tel: (965) 4843130, Fax: (965) 4842935, E-mail: atelfah@api.org.kw & atelfah@msn.com

Do Financial Planners Take Financial Crashes In Their Advice: Dynamic Asset Allocation under Thick Tails and Fast volatility Updating

Ahmad A. Telfah

Abstract

This paper is motivated by recent findings that current models of stock returns and interest rates are incapable to capture the actual behavior of those financial variables (like hyper volatility, the behavior of higher moments and leverage effect), particularly during rare events. The paper solves analytically for the optimal portfolio strategies of bonds, stocks and cash when the investment opportunity set is driven by a mixture of jump diffusion and non-affine stochastic processes of interest rates and stock returns. Such structure should be able to capture the characteristics of financial data during rare events as many recent articles indicate. Results show that investors hold a linear combination of a speculative portfolio and a hedging portfolio, with weights related to the investor's risk tolerance. The investor increases (decreases) the speculative allocation in his portfolio if he expects upward (downward) jump. The amount of increase or decrease to the speculative portfolio depends on the degree of risk aversion and the investment horizon. The hedging portfolio consists of additional bond portfolio to hedge against interest rate changes and additional stock portfolio to hedge against stochastic volatility changes. Those additional hedging portfolios depend on the duration of the bond and the correlation between stock returns and volatility processes, beside their dependence on the risk tolerance and investment horizon. The non-affine specification seems to increase the demand for hedging and captures the leverage effect. Calibration results show that the joint inclusion of jumps and non-affine structure into the investment opportunity set dynamics introduces a plausible simultaneous resolution for both Samuelson puzzle and asset allocation puzzle.

هل يأخذ المخططون الماليون الانهيارات المالية في اعتباراتهم: تنويع المحافظ المالية طويلة الأمد بافتراض التوزيع سميك الأطراف للعوائد والتقلبات السريعة في الذبذبة

ملخص

تأتي هذه الورقة مدفوعة بنتائج الدراسات الحديثة من أن النماذج الحالية المستخدمة في وصف سلوك العوائد على الأسهم وأسعار الفائدة لا تمثل بشكل مقنع السلوك الحقيقي لهذه المتغيرات المالية (مثل الذبذبة الجامحة، وسلوك العزوم العليا للتوزيع الإحصائي الذي تسلكه هذه المتغيرات وأثر الرفع المالي)، وخصوصا في الأحداث الطارئة والنادرة التي تصيب الأسواق المالية. وفي هذا الإطار تعمل الورقة على إيجاد التوزيع الأمثل في المحافظ المالية لكل من الأسهم والسندات والنقد المودع في البنوك عندما تحدد الحركة في العوائد على الأسهم وأسعار الفائدة التي تؤثر في منحنى الفرص الاستثمارية بمزيج من القفزات غير المنتظمة والمتقطعة والحركة العشوائية غير الخطية. إن مثل هذا التوصيف لسلوك العوائد على الأسهم وأسعار الفائدة يبدو أنه قادر على إعطاء صورة أقرب للسلوك الحقيقي للمتغيرات المالية التي تحدد منحنى الفرص الاستثمارية، خصوصا في الفترات التي تتعرض فيها الأسواق المالية لهزات نادرة وغير متكررة، كما تشير بعض الدراسات الحديثة. لقد أشارت النتائج أن المستثمر في العادة يحمل محفظة كلية تقسّم في الواقع إلى محفظتين جزئيتين حسب الغرض منها؛ المحفظة الأولى هي المحفظة الخاصة بالمضاربة والمحفظة الثانية هي تلك الخاصة بالتحوط، وبأوزان ترجيحية تتعلق بدرجة تقبل المخاطرة عند المستثمر. فالمستثمر يزيد الجزء المخصص للمضاربة من محفظته عندما يتوقع أن السوق المالي سوف تشهد قفزة للأعلى ويقلل هذا الجزء من المحفظة عندما يتوقع أن السوق سوف تشهد قفزة للأسفل. وحجم الزيادة و التقليل للجزء المخصص للمضاربة من المحفظة الكلية يتعلق مباشرة بدرجة تقبل المخاطرة وطول الأفق الاستثماري. أما بالنسبة للجزء المخصص للتحوط في المحفظة فإنه يحتوي على مقدار إضافي من السندات والأسهم وذلك بهدف التحوط ضد التغيرات في أسعار الفائدة والتقلبات في الذبذبة. أن مقدار ما يحمله المستثمر من السندات والأسهم في الجزء المخصص للتحوط يعتمد على أمد السند ومعاملات الارتباط بين العوائد ودرجة ذبذبتها هذا إلى جانب اعتمادها على درجة إقبال المستثمر على المخاطرة والأفق الاستثماري. أن استخدام التوصيف غير الخطي لحركة عوائد الأسهم وأسعار الفائدة زاد من الحصة المخصصة لأغراض التحوط في المحفظة، كما ساهم في ضبط أثر الرافعة المالية في النموذج. لقد دلت النتائج العملية لتطبيق النموذج أن أخذ القفزات غير المنتظمة إلى جانب البناء غير الخطي في الحسبان عند تمثل حركة الأسهم وأسعار الفائدة قد ساهم في حل معضلتي ساملسون Samuelson puzzle ومعضلة تخصيص الأصول المالية التي أشار إليها كنر ومانكو وويل (1997).

1. Introduction

Finance literature gives strong evidences that jumps in returns and stochastic volatility are both important components of index returns, especially in explaining the negative skewness and excess kurtosis in returns distribution[1]. However, recent findings conclude that models with jumps and the popular diffusive stochastic volatility structure as the one in Heston (1993) for example are incapable to capture the realistic behavior of returns, especially during event risks[2]. The source of the problem appears to be in the adopted form of the volatility process. These findings suggest that there is an additional, rapidly moving factor in volatility, which is persistent.

Bates (2000) and Pan (2002) suggest that there is evidence supports jumps in volatility as well. Mantegna and Stanley (1999) conclude that with the presence of thick tails and / or financial crashes volatility updates by faster than the square root of time, because thick tails contribute to volatility growth. They suggest that the exponential representation would be more appropriate. Jones (2003a) suggests that volatility should evolve according to a non-affine diffusion process. Specifically, he suggests using the constant elasticity of variance model (CEV) by replacing the square root in the variance diffusion term in Heston (1993) by an exponent of undermined magnitude as the non-affine interest rate process of Chan, Karolyi, Longstaff, and Sanders (1992) (CKLS thereafter). Jones (2003a) empirically conclude that a stochastic volatility model with a CEV specification reflects the hyper volatility updating at higher level of variance, and it captures as well the leverage effect. Jones (2003a) results confirm the conclusion of Mantegna and Stanley (1999) the and fast volatility updating during market crashes.

Same problems arise in the short rates. CKLS (1992), Das (2002), Jones (2003b) and Johannes (2004) and many others report the importance of non-affine structure and jumps in interest rates. Das (2002) describe the interest rate dynamics by affine mean-reversion model with jumps to capture three aspects of short rates dynamics; 1-the behavior of higher moments of the interest rate demonstrates considerable skewness and kurtosis, 2- it shows also persistent and high volatility of the short rates and 3- autocorrelation and mean reversion. CKLS (1992), Jones (2003b) and many others estimate a non-affine model for the short rates in both drift and diffusion and find that the non-affine specification of the diffusion term fits real data better than affine structures. As a matter of fact, CKLS (1992) reject all affine structures of the diffusion term in the interest rate models. Johannes (2004), on the other hand uses the non-affine stochastic interest rate with jumps to capture the three aspects that Das (2002) analyzes. Results of these papers generally indicate that jumps and non-affine structure of the interest rate diffusions are very important in modeling interest rates dynamics.

The suggestion of improving the volatility and interest rates models by including jumps has been used in asset pricing and it improved the performance of stochastic volatility and interest rates models for that purpose. Examples of using jumps in volatility in pricing

[1] See Bollerlev (1987), Bates (1996a&b), Duffie and Pan (1997), Das and Sundaram (1999), Lewis (2001) and many others.

[2] Including Bakshi, Cao and Chen (1997), Mantegna and Stanley (1999), Bates (2000), Benzoni (2001), Andersen, Benzoni, and Lund (2002), Pan (2002), Eraker, Johannes and Polson (2003) and Jones (2003a).

include Bakshi, Cao and Chen (1997 and 2000)[3], Scott (1997), Bates (2000), Duffie, Pan and Singleton (2000) and Pan (2002). In asset allocation, Liu, Longstaff, and Pan (2003) use the double jump model of Duffie, Pan and Singleton (2000) to study the implications of jumps in prices and volatility on optimal investment strategy. Examples of using jumps in pricing interest rate derivatives include Das and Frosti (1996) and Chacko and Das (2002). Those papers add the jump diffusion to Vasicek (1977) model.

The suggestion of using the non-affine structure in stochastic volatility or interest rates did not attract much work in asset pricing or asset allocation. In estimation, such models show good fit for real data. Jones (2003a), and Chacko and Viceira (2003a) estimate different non-affine stochastic volatility models and report exponent values that are significantly different from $1/2$ in Jones (2003a) and higher than 1 in Chacko and Viceira (2003a) almost for a ll frequencies. Similar results are reported with respect to the short rates, as in CKLS (1992), Jones (2003b) and Johannes (2004). Obviously, the reason for the lack of using non-affine processes in asset pricing and asset allocation is the intractability of those models, since such models do not usually give close form solutions.

This paper fills those gaps by analyzing the optimal portfolio choice when the investment opportunity set is driven by non-affine CEV stochastic interest rate with jump diffusions, and non-affine CEV stochastic volatility with jumps in stock returns index. For the short rates, the paper adopts a single factor model version of Johannes (2004) with differentiation between upward and downward jumps. The model basically adds jumps to the non-affine short rates process proposed by CKLS (1992). For stock returns, the paper uses the mixed non-affine stochastic volatility with jump process estimated by Chacko and Viceira (2003a). This process is basically the one estimated by Jones (2003a) but mixed with upward and downward jump diffusion processes. The use of the mixed non-affine structure with jumps in the short rates and stock returns is to account for the real aspects of financial data especially during financial crashes and explosions as mentioned in the above sited literature.[4].

Liu, Longstaff, and Pan (2003) test the impact of the corrective procedure that adds jumps to stochastic volatility in a dynamic asset allocation framework. This paper seeks to test the impact of the other suggested corrective procedure on the square root processes of stochastic volatility and short rates. It tests the impact of non-affine structure of stochastic volatility combined with jumps in return index on the optimal asset allocation. Chernov, Gallant, Ghysels and Tauchen (2003) suggest based on empirical estimation that affine models have to have jump in returns, stochastic volatility or both. The model used by Liu, Longstaff, and Pan (2003) is equivalent to the **AFF1V-JJ** model of Chernov *et al* (2003). The model used in this paper is equivalent to non-affine volatility version of **AFF1V-J** of Chernov *et al* (2003). Chernov *et al* (2003) did not discuss the non-affine volatility processes or the (CEV), leaving that to Jones (2003). In Chernov *et al* (2003) the model used in this paper can be described as **NON-AFF1V1r-J0J:** non-affine one stochastic volatility- one

[3] Bakshi, Cao and Chen (1997 and 2000) find that adding jumps to the square root stochastic volatility has a little impact on pricing or hedging long maturity options. They find that this innovation worsens the performance for short maturities.

[4] Early research by Bakshi, Cao and Chen (1997) and Scott (1997) use stochastic interest rate as a corrective procedure for the short fall of square root stochastic volatility process, but the procedure did not improve their results. I use the mixed non-affine stochastic interest rate with jumps to solve for the optimal portfolio choice of bonds, stocks and cash and not as a corrective procedure.

stochastic interest rate with jump in stock returns and jump in interest rates. This model can be extended easily to include jumps in volatility (**NON-AFF1V1r-JJJ***)* as shown later in the paper.

The paper basically analyzes the optimal portfolio mix of stocks, bonds and cash when market crashes (downward jumps) and market explosives (upward jumps) are possible. In analyzing that, the paper takes into account the hyper updating in volatility associated with such events in interest rates and stock index returns as well as the leverage effect. Jumps and stochastic volatility both allow for tail thickness in the stock return distribution. As Mantegna and Stanley (1999) suggest tail thickness is always associated with fast volatility updating. Additionally, at high level of volatility the negative correlation between the shocks in stock returns and shocks in volatility increases, and that strengthening the leverage effect as Jones (2003a) suggests. The same issue for the short rates, which displays high volatility and excess skewness and kurtosis that can be captured by the mixed CEV and jump model as suggested by Das (2002) and Johannes (2004).

As a starting point, I derived explicitly the zero coupon bond price contingents on the suggested non-affine CEV interest rate process. The closed form solution for the bond price is obtained by applying some kind of perturbation approximation methods of Kevorkian and Cole (1981). By deriving the zero coupon bond pricing formula, we can price all other derivative securities contingent on this bond like the European option, forwards and futures, swaps, caps, floors and European swaptions. I used the same method of approximation whenever needed in the paper to linearize the non-affine structure and non-linear terms. By means of this approximation, I get a linear closed form solution for the optimal portfolio strategy depends linearly on the model's parameters and the state variable (volatility). The optimal portfolio strategies are obtained without using any numerical techniques, even the standard finite difference techniques used to solve the non-linear expression of the optimal portfolios derived by Liu, Longstaff, and Pan (2003).

Results show that the optimal asset allocation is a linear combination of a speculative portfolio and hedging portfolio, weighted by the risk tolerance parameter (defined as a reciprocal of the relative risk aversion parameter). The demand of the speculative portfolio increases with the degree of risk tolerance, whereas the demand for the hedging portfolio decreases with risk tolerance. Although results indicate that investors are increasing or decreasing their speculative portfolio regarding to their expectation about upward and downward jumps, but it shows also that investors would increase their holdings during upward jumps to hedge the effect of downward jumps. The increase in allocation (during upward jumps) and the decrease in allocation (during downward jumps) depend crucially on the investment horizon and the risk aversion parameter.

The hedging portfolio on the other hand, consists of a hedging portfolio against stochastic interest rate and a hedging portfolio against stochastic volatility. Risk averse investor hedges interest rate risk by investing in bonds only. The size of this portfolio depends on the stochastic duration of the bond and the horizon investment, in addition to the degree of risk aversion. Investors also hedge stochastic volatility risk by investing in the stock index only, the size of this portfolio depends on the covariance between the stock returns and the volatility of stock returns, in addition to the investment time horizon and degree of risk aversion. The non-affine volatility structure seems to play very important role in hedging against volatility, through the correlation coefficient between shocks in stock returns and volatility shocks. This correlation increases with the level of current volatility causing the demand for hedging allocation to increase. The general result manifests clearly the effect of leverage on the hedging portfolio, where the negative correlation coefficient (negative skewness) increases at high levels of volatility inducing higher demand for hedging portfolio.

In the empirical part, the model is calibrated in two steps. In the first step the parameters of the model are estimated using monthly US data from April 1953 to September 2001 by means of the Spectral GMM techniques of Chacko and Viceira (2003). In the second step, the estimated parameters from the first step are used to calibrate the optimal portfolio choice for three different risk-recipients investors, with different investment horizons, aiming at mimicking the observed financial planners' advice. The calibration implemented by minimizing the sum of squared deviations between the theoretical and observed asset allocation advice across the different risk attitudes, different investment horizon investors. With this calibration, the model could provide simultaneous resolution for both the Samuelson and the asset allocation puzzle.

The literature that examines optimal asset allocation in dynamic setting is numerous. Campbell and Viceira (2002) survey most of the work that has been done in stochastic environment with no jump diffusion[5]. Studies that analyze asset allocation in event risk are limited, Liu, Longstaff, and Pan (2003) is the most significant paper, especially in analyzing the effect of double jumps in stock returns and stochastic volatility on asset allocation.[6]

In spite of Liu, Longstaff, and Pan (2003) and the overwhelming literature in asset allocation in stochastic environment, this paper is different and contributes to the literature in a very unique way. First by approximately deriving a close form solution for a price of a bond under a mixed non-affine CEV model with jumps that captures the behavior of higher moments and the high volatility of bond prices. Such models have been used extensively in estimation of the term structure of interest rate but not in pricing interest rate contingent derivatives. Secondly, by solving explicitly for the optimal portfolio mix of bonds and stocks under the assumption of double jump in asset returns and interest rates, taking into account two important aspects of the volatility of the stochastic variable in the presence of jumps. In one hand, the model captures the leverage effect and shows its impact on the hedging portfolio. On the other hand, it accounts for the fast updating in volatility and interest rate during market crashes by using a non-affine structure for both stochastic volatility interest rate processes and test its implication on optimal asset allocation strategies. Although, Liu, Longstaff, and Pan (2003) is not a special case of this model, the model can be extended as shown later in the paper to include jumps in volatility. In general, this paper fit into filling the gap of testing the corrective procedure to improve the stochastic volatility models performance [suggested by Mantegna and Stanley (1999) and Jones (2003b)] in asset pricing and asset allocation framework.

The paper is organized as follows, section 2 presents the formal model and the solution to the intertemporal portfolio problem. In section 3 we estimate capital market parameters and, subsequently, calibrate the model to observed asset allocation advice. Section 4 concludes.

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[5] Many papers have been published after their book, of which Bajeux-Besanainou, Jordan, and Portait (2002a&b), Liu (2001), and Munk et al (2004) and many others.

[6] Wu (2003) studies optimal portfolio choice in a stock return jump model, but he does not provide analytical solution.

2. Model Specifications

2.1 The Investment Opportunity Set Dynamics

The stock index is assumed to evolve according to the following set of mixed stochastic volatility stochastic interest rates jump diffusion differential equations

Where r_t is the short nominal interest rate, μ_S is the time varying expected excess

$$\frac{dS_t}{S_t} = (r_t + \mu_S)dt + \sqrt{v_t}dZ_S + \pi\sigma_r\sqrt{r_t}dZ_r + J_{Su}dN_u(\lambda_u) - J_{Sd}dN_d(\lambda_d) \tag{1}$$

$$dv_t = \kappa_v(\theta_v - v_t)dt + \sigma_v v_t^{\delta/2}dZ_v \tag{2}$$

$$dr_t = \kappa_r(\theta_r - r_t)dt + \sigma_r r_t^{\psi/2}dZ_r + J_{ru}dN_u(\lambda_u) - J_{rd}dN_d(\lambda_d) \tag{3}$$

return from investing in stocks, and v_t is the time varying stock index volatility. $Z_S, Z_r,$ and Z_v are Wiener processes. We assume that there is no correlation between the Brownian motions Z_S and Z_r or $Z_r,$ and Z_v. The instantaneous correlation between Z_S and Z_v is ρ_{Sv}.

Within this specification of stock returns, the shock to stock returns is a sum of two shocks, $\sqrt{v_t}dZ_S$ and $\pi\sigma_r\sqrt{r}dZ_r$[7], Accordingly, the volatility of stock returns V_S is $(v_t + \pi^2\sigma^2 r_t)$ and it depends on the v_t as well as the short rates[8].[9] The covariance between stock returns' shock and volatility shocks is time varying and given by $V_{Sv} = \sigma_v v_t^{\frac{\delta+1}{2}}\rho_{Sv}$. The covariance between stock returns' shock and the short rates shocks is also time varying and given by $V_{Sr} = \pi\sigma_r^2 r_t^{\frac{\psi+1}{2}}$.

$J_{Su}, J_{Sd}, J_{ru}, J_{rd} > 0$, are stochastic jump magnitudes. $dN_u(\lambda_u)$ and $dN_d(\lambda_d)$ defines an exponential upward and downward jump processes with jump frequencies or jump intensities λ_u, λ_d respectively. N_u and N_d are the exponential counting processes, and they represent the number of upward jumps and downward jumps up to time *t*, thus dN_u and dN_d represent incremental changes in *N* during an *infinitesimal* time period of length *dt*. λ_u, λ_d are positive constants:

$$dN_i = \begin{cases} 1 & \textit{with probability } \boldsymbol{\lambda}_i dt \\ 0 & \textit{with probability } (\mathbf{1} - \boldsymbol{\lambda}_i)dt \end{cases} \tag{4}$$

[7] Besanainou, Jordan, and Portait (2003a&b) and Liu (2001) use such specification in dynamic portfolio selection problems.

[8] Campbell (1987), Breen, Glosten and Jagannathan (1989), Shanken (1990), Glosten, Jagannathan and Runkle (1993) and Scruggs (1998) report the some empirical evidence that conditional volatility of stock returns depends on the short rates.

[9] Here there is a distinction in notation, v_t is used for the volatility process, V_S is the stock return volatility that depends on both v_t and r_t.

We assume that jump magnitudes are determined by draws from an exponential distribution with positive means η_u, η_d:

$$f(J_i) = \frac{1}{\eta_i} \exp\left(-\frac{J_i}{\eta_i} \right) \tag{5}$$

Where i is u, d. The upward and downward jumps are asymmetric. The jump size, J_i, does not depend on dt, instead, the probabilities (λ_u, λ_d) associated with the outcome are functions of dt. So as $\Delta t \to dt$, the jump size stays the same, but the jump probability decreases. This is a critical difference from the Brownian motions Z_S, Z_r, and Z_v where the increments become smaller as $\Delta t \to dt$. Chacko and Das (2002) and Chacko and Viceira (2003) differentiate between the upside and downside jumps.

The short rates and the volatility processes are described by a mean reverting non-affine stochastic processes. $\kappa_v, \theta_v, \kappa_r, \theta_r, \sigma_v, \pi, \sigma_r, \delta$ and ψ are constants. κ_v and κ_r represent the speed of adjustments for the volatility and short rates processes. θ_v and θ_r are the long term means for volatility and short rates, and σ_v and σ_r are the instantaneous volatility for the short rates and the volatility. When the constants δ and ψ take values greater than 1 ($\delta > 1$ and $\psi > 1$) the interest rates and the stochastic volatility are described as non-affine stochastic processes. Restricting δ and ψ to equal one results in the affine square root stochastic volatility process of Heston (1993) and the affine CIR interest rate process of Cox, Ingersoll and Ross (1985). The necessity of using the non-affine structure of stochastic volatility with jumps comes form Mantegna and Stanley (1999), and Jones (2003a) proposition regarding rapid volatility dynamics in the presence of jumps and to capture for the leverage effect. The use on the non-affine diffusion term of interest rate stems from Mantegna and Das (2002) and Jones (2003b) and Johannes (2004) suggestion of fast and persistent volatility of the short rates and the behavior of higher moments with the presence of jumps as recent empirical studies show.

Risks from different sources are priced in the stock excess return. The risk premium includes the prices of shocks and jumps risks:

$$\mu_S = \lambda_S v_t + \lambda_r \pi^2 \sigma^2 r_t + \lambda_u \eta_{Su} - \lambda_d \eta_{Sd} \tag{6}$$

Where λ_s and λ_r are the prices of the volatility and interest rates risks respectively, thus the instantaneous Sharp ratio (ISR) is defined by:

$$SR = \frac{\mu_S}{\sqrt{V_S}} = \frac{\lambda_S v_t + \lambda_r \pi^2 \sigma^2 r_t + \lambda_u \eta_{Su} - \lambda_d \eta_{Sd}}{\sqrt{(v_t + \pi^2 \sigma_r^2 r_t)}} \tag{7}$$

Equation (1) shows that the trajectories of the stock returns and the short rates consist of continuous path broken by occasional jumps with jump arrival intensities of λ_u and λ_d. In fact, the stochastic differential equations in (1) (even if we disregard stochastic volatility) is a mixed of normal process, Geometric Brownian Motion, (GBM), and Poisson-Exponential process in the jump part. This mixture result in an unknown conditional density function for

S_t and r_t. Additionally, with discretely sample data, it is difficult to know which returns have discontinuous components in it and which returns do not. The matter becomes more ambiguous when we add stochastic volatility, since it is unobservable stochastic variable, and also the density function is unknown (even with jumps exclusion).

Returns discontinuities typically exhibit themselves in discretely sampled data in the form of excess kurtosis[10]. So, one part of kurtosis in stock index returns can be captured by jump diffusions. The other part of kurtosis is captured by the difference between κ_v, σ_v. If the instantaneous volatility σ_v is large more volatile variance will lead to thick tails in the stock return distribution. Thus a mixed non-affine stochastic volatility jump diffusion process allows for complete kurtosis in stock returns[11].

Additionally, the dynamics in volatility are non-linear of the squared volatility, which makes volatility to updates faster with the presence of thick tails as Mantegna and Stanley (1999) suggest. The non-affine stochastic process also allows for leverage effect as Jones (2003) empirically concluded. Same analysis is applicable on the short rates.

Accordingly, the variation in the investment opportunity set is induced by stochastic variation of the short-term interest rate, jumps in the short term interest rates, stochastic variation of the expected excess return the stochastic conditional variance of stock returns and jumps probabilities in stock returns.

2.2 Bond Pricing

Proposition 1: Under the non-affine term structure of interest rates specified in (3), the approximate price of a zero-coupon bond with time to maturity τ is given by

$$B(r,t;\tau) = e^{-C(\tau)-A(\tau)r} \tag{8}$$

Where the values of $C(\tau)$ and $A(\tau)$ are given in the appendix, and the price of this bond evolves according to the following stochastic differential equation

$$\begin{aligned}\frac{dB_t}{B_t} &= (r_t + A(\tau)\lambda_r r_t^{\psi} + \lambda_u A(\tau)\eta_{ru} - \lambda_d A(\tau)\eta_{rd})dt + A(\tau)\sigma_r r_t^{\frac{\psi}{2}} dZ_{rt} \\ &\quad + J_{ru} A(\tau) dN_u(\lambda_u) - J_{rd} A(\tau) dN_d(\lambda_d)\end{aligned} \tag{9}$$

Proof: See Appendix.

Notice here that $A(\tau) = D(r,\tau) = -\dfrac{\partial B(r,\tau)}{\partial r_t}\dfrac{1}{B(r,\tau)}$ is the elasticity of the bond price with respect to the short interest rate; this elasticity is usually referred to as the stochastic duration of the interest rate contingent claim [Ingersoll, Skeldon and Weil (1978) and Cox, Ingersoll and Ross (1979)]. We assume that the bond available for the investor has a constant duration D. This can be thought of as reflecting the duration of the aggregate portfolio of bonds

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[10] The kurtosis for those distributions is $3 + \frac{1}{\eta}$.

[11] Bollerlev (1987) suggest that kurtosis in financial data is larger than what stochastic volatility produce. Andersen, Benzoni, and Lund (2002) reject the square root model of stochastic volatility for lack of kurtosis. Lewis (2001) considers that a process that combines stochastic volatility and jumps would produce enough tail thickness as financial data display.

outstanding, or a bond index, where bonds that expire are always substituted with new longer term bonds. Lets define $\sigma_B = \sigma_r D$, $\eta_{Bu} = \eta_{ru} D$, $\eta_{Bd} = \eta_{rd} D$, $J_{Bu} = J_{ru} D$, $J_{Bd} = J_{rd} D$ and $\lambda_B = \lambda_r D$.[12] To write the bond price dynamics in the following form:

$$\frac{dB_t}{B_t} = (r_t + \mu_B)dt + \sigma_B r_t^{\frac{\psi}{2}} dZ_{rt} + J_{Bu} dN_u(\lambda_u) - J_{Bd} dN_d(\lambda_d) \tag{10}$$

Where $\mu_B = \lambda_B r_t^{\psi} + \lambda_u \eta_{Bu} - \lambda_d \eta_{Bd}$. Also, note that the short interest rate and the return on the bonds are perfectly negatively correlated and with covariance

rate $V_{Br} = -\sigma_r \sigma_B r_t^{\psi} = -\left(\frac{1}{D}\right) V_B^2$ and that $\frac{\partial B}{\partial \lambda_u} < 0$, and $\frac{\partial B}{\partial \lambda_d} > 0$.

The comparative static means that upward jump frequencies of the short rate causes the bond price to fall, while the opposite happens with increasing the downward jump frequencies. Intuitively, as the upward jump frequency increases, the possibility of higher future rate increases and since the bond price is a discounted value of these rates the bond price decreases.

The parameter estimates of the CEV Jump diffusion model in equation (3) reported in Table 2 shows that the estimated long-term mean of the nominal interest rate is 3.42%, and the volatility of the volatility of interest rate is 2.44%. If you assume of the current level of the interest rate equals the long-term mean. Then with the estimated value of $\psi = 2.652$ we calculate the volatility of the interest rate to be 0.04 %. According to non-affine structure of the volatility of the interest rate increases as the level of the interest rate increases. The duration on a pure discount bond with 3 years to maturity $D(0,3)$ is $A(3)$ = 2.977 years. Its volatility (V_B) assuming the current level of the interest rate equal its long means would be $D(0,3) \times \sigma_r \times r^{\frac{\psi}{2}} = 0.08\%$, and this volatility increases at higher level of the interest rate.

2.3 Preferences

We consider the investment problem of an investor who has access to the capital market and wants to transfer current wealth W_0 into a future terminal wealth W_T at a specific investment horizon. We consider the basic asset allocation problem of how much to invest in a money market bank account (cash), nominal zero coupon bonds, and stocks. Here we assume that nominal pure discount bonds differ from cash in that they provide capital gains beside the interest. It is held also as a part of the hedging portfolio not only the speculation portfolio. The duration of the bond reflects the stochastic duration of the portfolio. Investors who hold bonds in their portfolio keep a certain portion of their portfolio in the form of bonds, whenever a bond expires it is replaced by a longer maturity bond.

[12] According to the specification in (1), the risk premium on the interest rates is $\frac{\lambda_r r_t^{\frac{\psi}{2}}}{\sigma_r}$, multiply this by the volatility of the bond $\sigma_r r_t^{\frac{\psi}{2}} D$, then $\lambda_B = \lambda_r D$

Accordingly, the investor is looking to choose a dynamic portfolio strategy to maximize the expected utility of terminal wealth at the horizon *T*. The utility function displays constant relative risk aversion (CRRA) utility function of the form:

$$u(W) = \begin{cases} \dfrac{W^{1-\gamma}}{1-\gamma}, & \text{if } W > 0 \\ -\infty, & \text{if } W \le 0 \end{cases} \tag{11}$$

Where $\gamma > 0$ is the risk aversion parameter. When $\gamma = 0$, we get the *log* utility function. The second part of the utility function imposes a condition of positive wealth at each time *t* from $t_0 < t < T$. At each period between $[t_0, T]$, the investor chooses the optimal allocation of stocks, bonds and cash in his portfolio, to invest α_S in stocks, α_B in the zero coupon bond and $(1 - \alpha_S - \alpha_B)$ in cash to maximize the expected utility of his terminal wealth W_T with respect to the fraction of wealth invested in risky assets

$$\underset{\alpha_S, \alpha_B}{Max}\, E[u(W_T)] \tag{12}$$

where the wealth process satisfying the self-financing condition

$$dW_t = \mu_W W dt + V_W W_t dZ_{Wt} + W\mathbf{J}_{\mathbf{Wu}} dN_u(\lambda_u) - W\mathbf{J}_{\mathbf{Wd}} dN_d(\lambda_d) \tag{13}$$

Where

$\mu_W = r + \boldsymbol{\alpha}'\boldsymbol{\mu}$, $V_W^2 = \boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}$, $\mathbf{J}_{\mathbf{Wu}} = \boldsymbol{\alpha}'\mathbf{J}_{\mathbf{u}}$ $\mathbf{J}_{\mathbf{Wd}} = \boldsymbol{\alpha}'\mathbf{J}_{\mathbf{d}}$

$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_S \\ \alpha_B \end{bmatrix}$, $\boldsymbol{\mu} = \begin{bmatrix} \mu_S \\ \mu_B \end{bmatrix}$, $\mathbf{J}_{\mathbf{u}} = \begin{bmatrix} J_{Su} \\ J_{Bu} \end{bmatrix}$, $\mathbf{J}_{\mathbf{d}} = \begin{bmatrix} J_{Sd} \\ J_{Bd} \end{bmatrix}$

and $\boldsymbol{\Sigma}$ is the variance covariance matrix $= \begin{bmatrix} V_S & V_{SB} \\ V_{BS} & V_B \end{bmatrix}$,where $V_{SB} = -DV_{Sr} = \pi\sigma_r\sigma_B r_t^{\frac{\psi}{2}}$

2.4 <u>The Optimal Asset Allocation Strategy</u>

Now define the value function or indirect utility function $U(W_t, r_t, v_t \tau)$ for an investor with τ periods investment horizon. The value function must satisfy the boundary condition $U(W, r, v, 0) = u(W)$. The Hamlton-Jacobian-Bellmen principle assumes that the value function at the maximum is martingale.

$$\underset{\boldsymbol{\alpha}'}{Max}\, E[dU] = 0 \tag{14}$$

$$
\begin{aligned}
\underset{\boldsymbol{\alpha}}{Max}\, 0 = \{ & \mu_W W U_W + \kappa_r(\theta_r - r_t)U_r + \kappa_v(\theta_v - v_t)U_v - U_\tau + \frac{1}{2}V_W^2 W^2 U_{WW} \\
& + \frac{1}{2}\sigma_r^2 r_t^{\psi} U_{rr} + \frac{1}{2}\sigma_v^2 v_t^{\delta} U_{vv} + V_{Wr} W U_{Wr} + V_{Wv} W U_{Wv} + V_{rv} U_{rv} \\
& + \lambda_u E\left[U(W + W\boldsymbol{\alpha}'\mathbf{J_u}, r_t + J_{ru}, v, \tau) - U(W, r_t, v, \tau)\right] \\
& - \lambda_d E\left[U(W - W\boldsymbol{\alpha}'\mathbf{J}_d, r_t - J_{rd}, v, \tau) - U(W, r_t, v, \tau)\right] \}
\end{aligned} \tag{15}
$$

Where $V_{Wv} = \boldsymbol{\alpha}'\begin{bmatrix} v_{Sv} \\ 0 \end{bmatrix}$, $V_{Wr} = \boldsymbol{\alpha}'\begin{bmatrix} V_{Sr} \\ V_{Br} \end{bmatrix} = \frac{1}{D}\boldsymbol{\alpha}'\begin{bmatrix} V_{SB} \\ V_B \end{bmatrix} = \frac{1}{D}\boldsymbol{\Sigma}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The partial differential equation (PDE) in (15) is highly non-linear in its arguments. To get an explicit solution for the optimal portfolio strategy, we have to solve PDE explicitly. The solution for the PDE in (15) is the value function or the indirect utility function $U(W_t, r_t, v_t \tau)$, with a terminal condition $U(W_t, r_t, v_t T - T) = u(W_T)$. To solve for the optimal portfolio selection, we first conjecture a solution for the value function that satisfy the terminal condition. Then substitute this conjecture and its derivatives in the PDE in (15). After that, we take the first order condition of the maximization with respect to the fraction of the wealth invested in the stock index and bonds ($\boldsymbol{\alpha}$). In the last stage, we verify that our conjecture is the correct form by substituting back the optimal portfolio policies into the PDE to solve explicitly for the parameters of the conjecture that satisfies the PDE above.

The first step to solve this equation, is to conjecture a solution in the form

$$
\frac{W^{1-\gamma}}{1-\gamma} f^{1-\gamma} \tag{16}
$$

and $f(r,, v_t, \tau)$ takes the form illustrated in (16) below:

$$
f(W, r, v_t, \tau) = Exp\left[a(\tau) + b(\tau)v + c(\tau)r\right] \tag{17}
$$

Where $a(0) = b(0) = c(0) = 0$, which satisfy the boundary condition for the value function $U(W, r, v, 0) = u(W)$. Now substitute the conjecture in the PDE in (15)

$$\underset{\boldsymbol{\alpha}}{Max}\, 0 = \left\{ \frac{1}{\gamma}\left(r_t + \boldsymbol{\alpha}'\boldsymbol{\mu}\right) + \frac{\kappa_r}{\gamma(1-\gamma)}(\theta_r - r_t)c(\tau) + \frac{\kappa_v}{\gamma(1-\gamma)}(\theta_v - v_t)b(\tau) \right.$$

$$- \frac{1}{\gamma(1-\gamma)}\left[\frac{da(\tau)}{d\tau} - \frac{db(\tau)}{d\tau} + \frac{dc(\tau)}{d\tau}\right] - \frac{1}{2}\left[\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}\right]$$

$$+ \frac{1}{2\gamma(1-\gamma)}\sigma_r r^{\psi} c^2(\tau) + \frac{1}{2\gamma(1-\gamma)}\sigma_r v^{\delta} b^2(\tau) \tag{18}$$

$$+ \frac{(1-\gamma)b(\tau)}{\gamma}\left[\boldsymbol{\alpha}'\mathbf{V}_{\mathbf{Wv}}\right] + \frac{(1-\gamma)c(\tau)}{\gamma}\left[\boldsymbol{\alpha}'\mathbf{V}_{\mathbf{Wr}}\right]$$

$$+ \lambda_u E\left[\left[\left(1+\boldsymbol{\alpha}'\mathbf{J}_{\mathbf{u}}\right)^{1-\gamma} e^{(1-\lambda)c(\tau)J_{ru}}\right] - 1\right]$$

$$\left. - \lambda_d E\left[\left[\left(1-\boldsymbol{\alpha}'\mathbf{J}_{\mathbf{d}}\right)^{1-\gamma} e^{(1-\lambda)c(\tau)J_{rd}}\right] - 1\right] \right\}$$

Lets take the terms $\left(1+\boldsymbol{\alpha}'\mathbf{J}_{\mathbf{u}}\right)^{1-\gamma}$ and $\left(1-\boldsymbol{\alpha}'\mathbf{J}_{\mathbf{d}}\right)^{1-\gamma}$ in the last two terms in the right hand side of (18). We know that $\left(1+\boldsymbol{\alpha}'\mathbf{J}_{\mathbf{u}}\right)^{1-\gamma} = (1+\alpha_S J_{Su} + \alpha_B J_{Bu})^{1-\gamma}$ and

$\left(1-\boldsymbol{\alpha}'\mathbf{J}_{\mathbf{u}}\right)^{1-\gamma} = (1-\alpha_S J_{Sd} - \alpha_B J_{Bd})^{1-\gamma}$. Assume that

$$M_u^{1-\gamma} = \left(1+\boldsymbol{\alpha}'\mathbf{J}_{\mathbf{u}}\right)^{1-\gamma} \tag{19}$$

$$M_d^{1-\gamma} = \left(1-\boldsymbol{\alpha}'\mathbf{J}_{\mathbf{d}}\right)^{1-\gamma} \tag{20}$$

Assume also that at a certain values of $\alpha_S, J_{Su}, \alpha_B$ and J_{Bu} there are m_u and m_d. $m_u > 1$ and $m_d < 1$ can be thought as previous values of M_u and M_d that are associated with a previous level of optimal asset of allocation in the stock index and the zero coupon bond during previous up and down jump experiences.

To linearize $M_u^{1-\gamma}$ and $M_d^{1-\gamma}$ we apply some kind of perturbation methods of Kevorkian and Cole (1981). This approximation is basically a Taylor series expansion around m_u and m_d. as follows:

$$M_u^{1-\gamma} \approx \gamma m_u^{1-\gamma} + (1-\gamma)m_u^{-\gamma} M_u \tag{21}$$

$$M_u^{1-\gamma} \approx \gamma m_u^{1-\gamma} + (1-\gamma)m_u^{-\gamma}(1+\boldsymbol{\alpha}'\mathbf{J}_{\mathbf{u}}) \tag{22}$$

and

$$M_d^{1-\gamma} \approx \gamma m_d^{1-\gamma} + (1-\gamma)m_d^{-\gamma} M_d \tag{23}$$

$$M_u^{1-\gamma} \approx \gamma m_d^{1-\gamma} + (1-\gamma) m_d^{-\gamma}(1+\boldsymbol{\alpha}'\mathbf{J_d}) \tag{24}$$

This approximation becomes crucial in deriving the optimal portfolio strategy, because it allows us to express the optimal allocation of bond and stocks explicitly with the state variables and the model parameters. Without such approximation, the optimal strategy will be a function of the optimal strategy, and the parameters of the value function will be also functions of the optimal allocation strategy. Liu, Longstaff, and Pan (2003) do not utilize such kind of approximation, so the optimal allocation of risky assets and the parameters of the value function they got depend directly on the optimal asset allocation, see their equations (17), (18), (21) and (28).

Accordingly, we can write the last two terms in (18) in the following way:

$$\left[\frac{1}{\gamma(1-\gamma)}\left[\gamma m_u^{1-\gamma} + (1-\gamma) m_u^{-\gamma}(1+\boldsymbol{\alpha}'\boldsymbol{\eta}_\mathbf{u})\right]\frac{\lambda_u}{1-(1-\gamma)c(\tau)\eta_{ru}} - \lambda_u\right] -$$
$$\left[\frac{1}{\gamma(1-\gamma)}\left[\gamma m_d^{1-\gamma} + (1-\gamma) m_d^{-\gamma}(1+\boldsymbol{\alpha}'\boldsymbol{\eta}_\mathbf{d})\right]\frac{\lambda_d}{1+(1-\gamma)c(\tau)\eta_{rd}} - \lambda_d\right]$$

Where $E[j_i] = \eta_i$, $\boldsymbol{\eta}_\mathbf{i} = \begin{bmatrix}\eta_{Si}\\ \eta_{Bi}\end{bmatrix}$ and $i = u, d$. And $E\left[e^{(1-\lambda)c(\tau)J_{ru}}\right] = \frac{1}{1-(1-\gamma)c(\tau)\eta_{ru\ i}}$ and $E\left[e^{(1-\lambda)c(\tau)J_{rd}}\right] = \frac{1}{1+(1-\gamma)c(\tau)\eta_{rd}}$.

The first order condition of equation (18) with respect to $\boldsymbol{\alpha}$ results in the solution for the optimal asset allocation for an investor maximizes CRRA utility function in wealth, subject to the wealth dynamics shown in equation (13).

Proposition 2: The optimal portfolio weights for investor with CRRA utility function who invest in stock index, zero coupon bond and cash for τ ***periods investment horizon and subject to the constraint in (13) is:***

$$\boldsymbol{\alpha} = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{\mu} + q_u(\gamma,\tau)\boldsymbol{\eta}_\mathbf{u} - q_d(\gamma,\tau)\boldsymbol{\eta}_d\right] - \left(1-\frac{1}{\gamma}\right)\left[\left(\frac{c(\tau)}{D}\right)\begin{bmatrix}0\\1\end{bmatrix} + \boldsymbol{\Sigma}^{-1}b(\tau)V_{Sv}\begin{bmatrix}1\\0\end{bmatrix}\right] \tag{25}$$

The residual $\alpha_{r.} = 1 - \mathbf{1}'\boldsymbol{\alpha} = 1-\alpha_B - \alpha_B$ ***is invested in the cash or bank account.***

$q_u(\gamma,\tau) = \frac{\lambda_u m_u^{-\gamma}}{1-(1-\gamma)c(\tau)\eta_{ru}}$, *and* $q_d(\gamma,\tau)\frac{\lambda_d m_d^{-\gamma}}{1+(1-\gamma)c(\tau)\eta_{rd}}$.

Where $b(\tau)$ ***and*** $c(\tau)$ ***are the solutions to the following Riccati ordinary differential equations (Riccati ODE's):***

$$\frac{db(\tau)}{d\tau} + l_0 + l_1 b(\tau) + \frac{1}{2} l_2 b^2(\tau) = 0 \tag{26}$$

$$\frac{dc(\tau)}{d\tau} + h_0 + h_1 c(\tau) + \frac{1}{2} h_2 c^2(\tau) = 0 \tag{27}$$

Where $l_0, l_1, l_2, h_0, h_1, h_2, c(\tau),$ and $b(\tau)$ ***are provided in the appendix.***

Proof, see appendix.

Equation (25) shows that the optimal portfolio weights for CRRA investors are linear combinations of a speculative portfolio (the first 3 terms of equation 25) and a hedge portfolio against changes in short rates and volatility. The allocation in the speculative portfolio decreases (increases) with the risk aversion (tolerance) parameter $\gamma\left(\frac{1}{\gamma}\right)$, whereas the allocation in the hedge portfolio increases (decreases) with risk aversion (tolerance) parameter $\gamma\left(\frac{1}{\gamma}\right)$. In particular, for investors with the same investment horizon τ the optimal portfolios are linear combinations of the speculative portfolio and a single hedge portfolio; the relative risk tolerance $1/\gamma$, represents the weights on the two relevant portfolios.

The speculative portfolio includes the usual speculative portfolio (the first term) that is optimal for an investor with short horizon or log utility (myopic investor) and a speculative allocation related to the jump risk. The myopic portfolio is a mixed of bonds and stocks. The optimal mixture of bonds and stocks in this portfolio does not depend on risk aversion parameter or investment horizon, it depends entirely on the expected returns and variances of bonds and stock. If we assume pure diffusion process of asset returns (disregard all other terms in equation (25) and look at the first term only), we find that investor with high risk aversion parameter would invest less in this portfolio and more in cash and visa versa. The risk aversion parameter has no impact on the components of the myopic portfolio. This is consistent with the strong 2-fund separation theorem of Cass and Stiglitiz (1970).

Before analyzing the inclusion of jumps, lets see how the values of $q_u(\gamma,\tau)$ and $q_d(\gamma,\tau)$ by the constant risk aversion parameter and investment horizon. Differentiating $q_u(\gamma,\tau)$ and $q_d(\gamma,\tau)$ with respect to γ and τ results in the following comparative static results:

$$\frac{\partial q_u(\gamma,\tau)}{\gamma} < 0, \; \frac{\partial q_u(\gamma,\tau)}{\tau} < 0 \tag{28}$$

$$\frac{\partial q_d(\gamma,\tau)}{\gamma} > 0, \; \frac{\partial q_d(\gamma,\tau)}{\tau} > 0 \tag{29}$$

Equation (25) indicates that risk averse investors (regardless of the degree of their risk tolerance and their investment horizon) will increase their speculative allocation when the market explodes (upward jump) and decrease their speculative allocation when the market crashes (downward jump). However, the size of the increase and decrease in the speculative portfolio depends basically on the degree of risk aversion and investment horizon.

Comparative static implies generally that short-term investors (regardless of the degree of their risk aversion) will increase their speculative allocation by substantial amount when there is upward jumps ($q_u(\gamma,\tau)\,\boldsymbol{\eta}_{\mathbf{u}}$ will be large) and decrease their speculative allocation by small amount when there is downward jumps ($q_d(\gamma,\tau)\,\boldsymbol{\eta}_{\mathbf{d}}$ will be small). Long-term investors will behave in the opposite way, they will increase their speculative allocation by small amount during upward jumps fearing a future downward jump and reduce their speculative portfolio by significant amount during downward jumps.

It implies also that aggressive investor (investor with low γ) will increase his speculative portfolio by significant amount during upward jumps and reduce his speculative portfolio by small amount when there is downward jump. For this investor, the positive marginal utility from gain is higher than the negative marginal utility from loss. On the other hand that conservative investor (investor with high γ) will increase his speculative portfolio by small amount during upward jumps and reduce his speculative portfolio by significant amount when there is downward jump. For this investor, the negative marginal utility from losses is higher than the positive marginal utility from gains.

Accordingly, if there is a positive jump in interest rate or stock return index or both of them ($\lambda_u > 0, \eta_{ru} \geq 0,$ and $\boldsymbol{\eta}_{\mathbf{u}} > 0$), short-term aggressive investor (investor with low γ and τ) would utilize this upward jumps by significantly increase his speculative portfolio allocation ($q_u(\gamma,\tau)\,\boldsymbol{\eta}_{\mathbf{u}}$ will be large). While the same investor will reduce his speculative portfolio by a small amount when there is down ward jump. No surprise he is aggressive investor looking for fast speculative profits and fast gain affects his utility more than loss does. On the other hand long term conservative investor (investor with high γ and τ) would increase his speculative portfolio allocation with small amount ($q_u(\gamma,\tau)\,\boldsymbol{\eta}_{\mathbf{u}}$ will be small) fearing from future downward jumps. Of course the same investor will reduce his speculative portfolio substantially when there is down ward jump.

The changes in the speculative portfolio allocation depend on a previous allocation level during previous upward and downward jump. $q_u(\gamma,\tau)$ and $q_d(\gamma,\tau)$ does not imply any changes in the composition of the myopic portfolio based on risk aversion or investment horizon. As long as the probability of upward and downward jump arrivals in stock returns or short rates is the same, the composition of the speculative portfolios will not change. What might affect the composition is the expected value of the jump sizes (the $\boldsymbol{\eta}$ vectors). Generally, we can consider $q_u(\gamma,\tau)$ as a hedge portfolio against downward jumps, since the comparative static shows that this factor is negatively related to risk aversion parameter γ . Conservative investors hold more of this portfolio whereas aggressive investors hold less of it.

Liu, Logstaff and Pan (2003) derive optimal asset allocation under double jump diffusion process in both stock prices and volatility[13]. The optimal asset allocation they derived is non-linear in the optimal asset allocation itself and the parameter $c(\tau)$, (B in their

[13] Including upward and downward jump in volatility as in Liu, Logstaff and Pan (2003) makes $q_u(\gamma,\tau)=\frac{\lambda_u m_u^{-\gamma}}{1-(1-\gamma)\left(c(\tau)\eta_{ru}+b(\tau)\eta_{vu}\right)}$,and $q_d(\gamma,\tau)\frac{\lambda_d m_d^{-\gamma}}{1+(1-\gamma)\left(c(\tau)\eta_{rd}+b(\tau)\eta_{vd}\right)}$

paper), where B itself also depends on the optimal asset allocation. To solve for the optimal allocation in their model, you need to use some kind of numerical finite difference techniques. Their model does not specify different jump intensities for upward and downward jumps thus jumps in their model appear in the speculative portfolio, with no hedging components.

The hedging portfolio in equation (25) (the last two terms) describes how the investor optimally hedges changes in the opportunity set. The first term in the hedging portfolio describes the hedge against the nominal interest rate and it summarizes the investor's attitude towards changes in the interest rate. The optimal hedge against changes in the interest rate is obtained by investing entirely in the bond. Hedging against stock return volatility dynamics is described by the last term in equation (25). The optimal hedge against volatility dynamics is obtained by entirely investing in the stock index.

As shown earlier, the size of the hedging portfolio depends on the relative risk aversion parameter. Aggressive investors invest less in the hedging portfolio and more in the speculative portfolio, whereas conservative investors hold more from the hedging portfolio and less from the speculative portfolio. Hedging portfolio depends also on the investment time horizon through the parameters $c(\tau)$ and $b(\tau)$. Hedging against interest rate changes depends crucially on the duration of the zero coupon bond, whereas hedging against the stock return volatility dynamics depends on the covariance between stock returns and the volatility of stock returns.

Brennan and Xia (2000), Bajeux-Bensnainou, Jordan and Protait (2002a&b) Munk, Sørensen, and Vinther (2004) got a hedging term against interest rate changes in a stocks-bonds portfolio mix with Vasicek mean reverting process that have the same impact on the allocation as the one derived in equation (25). Chacko and Viceira (2002) and longstaff (2001) study hedging against stochastic volatility. Liu (2001) derives an optimal portfolio hedging against short rates dynamic and stochastic volatility in square root process for volatility and CIR model of the short rates, however, his model does not show that the demand on those hedging portfolios increase with risk aversion parameter.

The optimal asset allocation strategy in equation (25) shows explicitly that the bond-stocks- cash mix can be changed among investors with respect to their risk aversion and their investment horizon. Investors may reallocate their speculation portfolio if they expect an up or down jump in the stock returns or interest rate or both of them and the additional positive or negative holdings depends on the risk aversion parameter and the investment horizon length in the way explained above. Additionally, investors will hold a separate bond portfolio and a separate stock portfolio to hedge against stochastic changes in interest rate and stock prices respectively. The sizes of those portfolios depend on the risk aversion parameter and the investment horizon. Accordingly, an expression as the one in (25) that consists mixed positions in e stocks and bonds can introduce simultaneous resolution for both the Samuelson puzzle and the asset allocation puzzle of Canner, Mankiw, and Weil (1997).

2.5 The Effect of Non-Affine Structure

Equation (25) shows that hedging against volatility dynamics depends crucially on the covariance between the shocks in the stock index returns and the shocks in volatility. This covariance contains the effect of the negative skewness and kurtosis that is captured totally by ρ_{Sv} and σ_v respectively. The correlation coefficient between the shocks in stock returns and the shocks in volatility is $\frac{\sigma_v v_t^{\frac{\delta}{2}}}{\sqrt{(v_t + \pi^2 \sigma^2 r_t)}}$. The effect of non-affine structure of the stochastic volatility[14] appears clearly in the correlation coefficient, which strengthen the so-called the leverage effect. The negative correlation between stock returns and volatility increases at higher levels of volatility.

At high levels of volatility, the non-affine structure will result in hyper increase in volatility and hence an increase in the correlation coefficient. When volatility exceeds100 percent[15], with $\delta > 2$ volatility updates faster and the correlation coefficient increases implies more negative skewness (leverage effect) and kurtosis. On the other hand, at low level of volatility, a value of $\delta > 2$ makes volatility decays overtime, and the negative correlation between stock returns and volatility decreases. Thus, the non-affine structure captures the leverage effect of stock returns where the negative skewness of stock returns increases at higher volatility levels and decreases at lower volatility levels as Jones (2003) suggests.

Higher volatility and correlation imply higher demand on the hedging portfolio of volatility dynamics, and lower volatility and hence lower correlation coefficient imply lower demand on the volatility-hedging portfolio. Thus, the demand on the volatility-hedging portfolio depends on the volatility state variables, which means that investors time the market when they constitute their hedging portfolios. This provides significant difference from the square root volatility models where the correlation coefficient is not time varying and the demand on the volatility hedging portfolio does not depend on the state variable, so there is no market timing in stochastic volatility.

The non-affine structure of the short rates affects both the optimal demands for hedging against stochastic volatility and stochastic interest rate. It affect the demand for stochastic volatility hedging portfolio through the inverse of the variance covariance matrix, and it affects the demand on the interest rate hedging portfolio through the duration of the bond.

3. Model Estimations and Calibrations

In the following three subsections we will first introduce the estimation technique spectral GMM of Chacko and Viceira (2003a) and Singleton (2001). Then we are going to calibrate the asset prices and volatility. parameters of the capital market model. In the third section, we use these parameters in a calibration exercise where the subjective risk aversion parameter and time horizon parameter are fitted to match observed asset allocation advice for different investor groups

3.1 Model Estimation: Spectral GMM

The paper adopts the spectral GMM approach to estimate the parameters of the processes in the model Pennacchi (1991), Campbell and Viceira (2001) and Brennan and Xia (2002) have used Kalman filtering in contexts similar to this paper. Chacko and Viceira (2003a&b) use the spectral GMM in stochastic volatility context. One advantage of the

[14] The inverse of the variance covariance matrix $\mathbf{\Sigma}^{-1}$ does not depend on δ the power of volatility.

[15] VIX index of the S&P 500 stock index option implied volatility increased by 313 percent on October 19, 1987, 53 percent on October 27, 1997, and 28 percent on August 27, 1998.

spectral GMM over the Kalman filtering is that the spectral GMM does not require the discretization of the stochastic process. It only requires knowledge of its conditional characteristic function. Once we know this function, we can integrate the stochastic interest rate and inflation out and obtain the characteristic function of next period's stock price and commodity price level conditional only on the prior period's prices. Chacko and Viceira (2003a) call this estimation method Spectral GMM because we can use generalized method of moments (GMM) to estimate the parameters of the model directly off this conditional characteristic function.

To calculate the conditional characteristic function, we have to transform the process to an exponential affine process as in Chacko and Das (2002). We transform the stock price process into a form of the log stock price, and then we apply the following steps:

Deriving the ccf will be implemented according to the following steps[16]:

1. Deriving the Kolomogorov Backward Equation (KBE) or Fokker-Plank Forward Equation (F-PFE), two names for same equation. The KBE or the F-PFE is a partial differential equation with a known solution form. The conditional characteristic function is the solution for that equation. And this whole procedure is known as Feynman-Kac Formula.
2. To solve KBE we conjecture a solution for the characteristic function and substitute this conjecture into the KBE.
3. When substituting the conjecture into the KBE, we get two ordinary differential equations (ODE) of the form of Raccati equations.
4. Solving those two Raccati equations gives the parameters of the characteristic function.

3.2 Estimating the Parameters of the Processes

The system is estimated using monthly US data with almost 50 years period from April 1953 until December 2003. Data on seven constant maturity yields are used; the times to maturities are 3 months, 1-year, 2 years, 3 years, 5 years, 10 years, and 20 years. Unavailable yields are calculated using the simple bootstrap method. Cumulative dividend stock returns data available in Robert Shiller’s web site are used for the purpose of estimating the stock returns process. Table 1 shows the data used and the sources of these data.

Table 2 shows the estimation of the parameter of the investment opportunity set. The most important thing in this estimation is the values of the exponents δ and ψ. ψ is significantly different from 1 but δ is not significantly distinguishable from 1. The correlation coefficient estimate ρ_{SV} = - 0.31 which implies the negative skewness of stock returns. At a current level of volatility equals the long term mean of volatility the covariance between the stock returns and the volatility equals –20%. Chacko and Viciera (2003) report same results for the stock return dynamics with non-affine stochastic volatility. Bond price parameters are discussed earlier in section 2.2, Munk et al (2004), and Campbell and Viceira

[16] Chacko and Viceira (2003a) has the full description. The characteristic function derivations for the processes used in this paper are available by the author.

3.3 Calibration to The Professional Financial Planners' Advice

In this part, we follow Munk et al, (2004) exercise in trying to match the financial planners' advice. However, the matching data here is constructed in different way.

Table 3 tabulates the asset allocation recommendations as considered the match data. These recommendations are generated from the advice of the four financial planners and their three classic portfolios and rank them by their market risk: low risk, medium risk, and high risk. These portfolios are tabulated in Canner, Mankiw and Weil (1997) as for "conservative," "moderate," and "aggressive" investors. We construct those portfolios in a way that the most conservative advice is assigned to the short horizon conservative investor, and the least conservative advice is assigned to the long horizon conservative investor. On the other hand, the most aggressive advice is assigned to the long run aggressive investor, and the least aggressive advice is assigned to the short-term aggressive investor. So, the short run horizon takes the most conservative or the least aggressive of all advices, and the long run horizon takes the least conservative or the most aggressive advices. The medium horizon takes the medium of the all advices.

The recommendations in Table 3 are in accordance with the popular advice that investors with a long horizon should invest a higher fraction of wealth in stocks. Also, the investment recommendations are in accordance with the pattern pointed out by Canner, Mankiw and Weil (1997) and, in fact, for any investment horizon the bond to stock ratio is increasing with risk aversion.

We calibrate parameters so as to minimize the sum of squared deviations between the asset allocation recommendations in Table 3 and the optimal asset allocations in the economic modeling framework in section 2. The summation of squared deviations that will be minimized is taken over all portfolio weights for the three horizons (short, medium, long), the three degrees of risk aversion (conservative, moderate, aggressive), as well as the allocations into stocks, bonds, and cash. This makes a total of 27 (= 3 x 3 x 3) squared deviations in the summation.

In calibrating the model, we vary three risk aversion parameters: $\gamma_{\mathrm{con}} > \gamma_{mod} > \gamma_{agg} > 0$. Likewise, we vary three investment horizon parameters: $0 < H_{short} < H_{med} < H_{long} < 35$ years. These parameters are meant to represent the relative risk aversion parameters of "conservative," "moderate," and "aggressive" investors as well as the investment horizon of investors with short, medium, and long horizons, respectively.

Furthermore, we allow investors with different investment horizons to use bonds that differ in duration. The individual investor can thus invest in cash, stocks and a bond with a duration that depends on the investment horizon. Without loss of generality the bond can be thought of as a zero coupon bond and when we refer to the duration of the bond in the following, we are in fact referring to the time to maturity on the relevant zero coupon bond. This duration concept is known as the stochastic duration as shown by Ingersoll, Skeldon and Weil (1978) and Cox, Ingersoll and Ross (1979). We calibrate the stochastic durations as part of the problem and we impose the intuitive restriction that investors with longer investment horizons should not use shorter duration bonds and the restriction that the duration on the bond is between 5 years and 15 years so that it could represent a realistic aggregate bond index; i.e. in the calibration we have the restriction: $5 \le D_{short} \le D_{med} \le D_{short} \le 15$.

We perform one calibration by varying only the risk attitude parameters, investment horizons, and relevant durations subject to the above restrictions. The point estimates of the asset price and interest rate in Table 2 are applied in generating the optimal theoretical asset allocations as we derived in equation (25).

It can be observed that the calibrated model asset allocation in Table 4 conforms to the advice that longer term investors should invest a higher fraction of wealth in stocks. It confirms also the advice that aggressive investors should allocate more stocks in their portfolios as compared with bonds. The trend in table 4 is quite obvious. The short horizon-conservative investors allocate 1.125 % in bonds relative to stocks, which is a little less than the financial planner advice for the short-term – conservative investors. However, our model could not mimic exactly the advice for the long-term- aggressive investor. According to our calibrations, this investor's bonds to stocks ratio is 6%, where it is zero for the match data. Generally speaking, the model and the estimates could mimic closely the financial planner advice.

The representative investment horizons calibrated in Table 4 - B seems to be reasonable. Specifically, investor with a short investment horizon has an investment horizon of 5.36 years while a long-term investor acts so as to maximize utility of wealth at a thirty five years horizon. On the other hand, the calibrated relative risk aversion parameters are seemed to be high. For example, an "aggressive" investor has a relative risk tolerance of 0.62 (= 1/1.62). Hence, this investor will only allocate 62% of wealth to the speculative portfolio while the remaining 38% is allocated to the hedge portfolio. "Conservative" investors on the other hand only allocate 10% (= 1/10.26) of wealth to the speculative portfolio while 90% are invested in a hedge portfolio.

4. Conclusion

The paper analyzes the optimal portfolio mix of stocks, bonds and cash when market crashes (downward jumps) and market explosives (upward jumps) are possible. In analyzing that, the paper takes into account the hyper updating in volatility associated with such events in interest rates and stock index returns as well as the leverage effect. Jumps and stochastic volatility both allow for tail thickness in the stock return distribution. Mantegna and Stanley (1999) suggest that tail thickness is always associated with fast volatility updating. Additionally, at high level of volatility the negative correlation between the shocks in stock returns and shocks in volatility increases, and that strengthening the leverage effect as Jones (2003a) suggests. Short rates also displays high volatility and excess skewness and kurtosis that can be captured by the mixed CEV and jump model as suggested by Das (2002) and Johannes (2004).

The paper utilizes a perturbation approximation method used Kevorkian and Cole (1981) to derive closed form pricing formula for a zero coupon bond pricing. The same approximation is used to derive explicitly linear optimal portfolio strategy for stocks and bonds. Based on such approximation methods, we can price all other derivative securities contingent on this bond like the European option, forwards and futures, swaps, caps, floors and European swaptions. By the means of those approximation techniques, we can price explicitly different stock contingent securities without going for numerical techniques.

Results show that the optimal asset allocation is a linear combination of a speculative portfolio and hedging portfolio, weighted by the risk tolerance parameter (defined as a reciprocal of the relative risk aversion parameter). The demand of the speculative portfolio increases with the degree of risk tolerance, whereas the demand for the hedging portfolio decreases with risk tolerance. Although results indicate that investors are increasing or decreasing their speculative portfolio regarding to their expectation about upward and downward jumps, but it shows also that investors would increase their holdings during upward jumps to hedge the effect of downward jumps. The increase in allocation (during upward jumps) and the decrease in allocation (during downward jumps) depend crucially on the investment horizon and the risk aversion parameter.

The hedging portfolio on the other hand, consists of a hedging portfolio against stochastic interest rate and a hedging portfolio against stochastic volatility. Risk averse investor hedges interest rate risk by investing in bonds only. The size of this portfolio depends on the stochastic duration of the bond and the horizon investment, in addition to the degree of risk aversion. Investors also hedge stochastic volatility risk by investing in the stock index only, the size of this portfolio depends on the covariance between the stock returns and the volatility of stock returns, in addition to the investment time horizon and degree of risk aversion. The non-affine volatility structure plays very important role in hedging against volatility, through the correlation coefficient between shocks in stock returns and volatility shocks. The correlation increases with the level of current volatility causing the demand for hedging allocation to increase. The general result indicates clearly the effect of leverage through the hedging portfolio, where the negative correlation coefficient (negative skewness) increases at high levels of volatility inducing higher demand for hedging portfolio.

The calibration of the model on US monthly data shows that investors with different risk tolerance and different time horizon would allocate differently in the presence of the jumping and stochastic investment opportunity set. Those results provide a simultaneous resolution for both the Samuelson and the asset allocation puzzle.

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Appendix

Appendix

Proof of Proposition 1: The zero coupon bond price formula:
The interest rate process under the risk neutral measure Q is

$$dr_t = [\kappa_r\theta_r - \lambda_u\eta_u + \lambda_d\eta_d - \kappa_r r_t - \lambda r^{\psi}]dt + \sigma_r r_t^{\psi/2} dZ_r^Q + J_{ru} dN_u^Q(\lambda_u) - J_{rd} dN_d^Q(\lambda_d) \quad \text{(A1)}$$

Using the pricing partial differential difference equation (PDDE) for the price of the bond as in Black and Scholes (1973), Merton (1973) Courtadon (1982), Cox, Ingersoll and Ross (1985), Ahn and Gao (1999) and Chacko and Das (2002).

$$\begin{aligned}&[\kappa_r\theta_r - \lambda_u\eta_u + \lambda_d\eta_d - \kappa_r r_t - \lambda r^{\psi}]B_r + \frac{1}{2}\sigma_r r^{\psi} B_{rr} + B_\tau + \\ &+ \lambda_u E\big[B(r_t + J_{ru},\tau) - B(r_t,\tau)\big] - \lambda_d E\big[B(r_t - J_{ru},\tau) - B(r_t,\tau)\big] = -r_t B\end{aligned} \quad \text{(A2)}$$

Under the boundary condition $B(r_t,0) = 1$, the solution of the PDDE above is of the form

$$B(r_t,\tau) = Exp[-A(\tau) - C(\tau)] \quad \text{(A3)}$$

Now substituting the conjecture and its derivatives in (A2) we get:

$$\begin{aligned}&-[\kappa_r\theta_r - \lambda_u\eta_u + \lambda_d\eta_d - \kappa_r r_t - \lambda r^{\psi}]A(\tau) + \frac{1}{2}\sigma_r r^{\psi} A^2(\tau) - \frac{dA(\tau)}{d\tau} r_\tau - \frac{dC(\tau)}{d\tau} \\ &+ \lambda_u E\big[Exp(AJ_{ru}) - 1\big] - \lambda_d E\big[Exp(-AJ_{rd}) - 1\big] = -r_t\end{aligned}$$

Now, we need to linearize the term r^{ψ}

$$r_t^{\psi} \approx (1-\psi)\theta_r^{\psi} + \psi\theta_r^{\psi-1} r_t \quad \text{(A4)}$$

$$E\big[Exp(AJ_{ru})\big] = \frac{1}{1 - A\eta_{ru}} \quad \text{(A5)}$$

$$E\big[Exp(-AJ_{rd})\big] = \frac{1}{1 + A\eta_{rd}} \quad \text{(A6)}$$

$$\begin{aligned}&\left[\left(\frac{\lambda_u}{1 - A\eta_u} + \frac{\lambda_d}{1 - A\eta_d}\right) - (\lambda_u + \lambda_d) + r_t\right] - [\kappa_r\theta_r - \lambda_u\eta_u + \lambda_d\eta_d - \lambda_r\theta_r^{\psi}(1-\psi) - (\kappa_r + \lambda_r\theta_r^{\psi-1}\psi)r_t]A(\tau) \\ &+ \frac{1}{2}\big[\sigma_r^t\theta_r^{\psi}(1-\psi) + \sigma_r^t\theta_r^{\psi-1}\psi r_t\big]A^2(\tau) - \frac{dA(\tau)}{d\tau} r_\tau - \frac{dC(\tau)}{d\tau} = 0\end{aligned} \quad \text{(A7)}$$

$$c + aA(\tau) + \frac{1}{2}bA^2(\tau) - \frac{dA(\tau)}{d\tau}r_\tau - \frac{dC(\tau)}{d\tau} = 0 \quad \text{(A8)}$$

$$a = a_1 + a_2 r_t \quad \text{(A9)}$$

$$b = b_1 + b_2 r_t \quad \text{(A10)}$$

$$c = c_1 + c_2 r_t \quad \text{(A11)}$$

Where

$$a_1 = [\kappa_r\theta_r - \lambda_u\eta_u + \lambda_d\eta_d - \lambda_r\theta_r^{\psi}(1-\psi)]$$
$$a_2 = -\left(\kappa_r + \lambda_r\theta_r^{\psi-1}\psi\right)$$
$$b_1 = \sigma_r^2\theta_r^{\psi}(1-\psi)$$
$$b_2 = \sigma_r^t\theta_r^{\psi-1}\psi$$
$$c_1 = \left(\frac{\lambda_u}{1-A\eta_u} + \frac{\lambda_d}{1-A\eta_d}\right) - \left(\lambda_u + \lambda_d\right)$$
$$c_2 = 1$$

$$\left(c_1 + a_1A(\tau) + \frac{b_1}{2}A^2(\tau) - \frac{dC(\tau)}{d\tau}\right) + \left(c_2 + a_2A(\tau) + \frac{b_2}{2}A^2(\tau) - \frac{dA(\tau)}{d\tau}\right) = 0 \quad \text{(A12)}$$

$$\left(c_1 + a_1A(\tau) + \frac{b_1}{2}A^2(\tau) - \frac{dC(\tau)}{d\tau}\right) = \left(c_2 + a_2A(\tau) + \frac{b_2}{2}A^2(\tau) - \frac{dA(\tau)}{d\tau}\right) = 0 \quad \text{(A13)}$$

Equation (A13) can be separated into two ordinary differential Raccati equations as follows:

$$c_1 + a_1A(\tau) + \frac{b_1}{2}A^2(\tau) = \frac{dC(\tau)}{d\tau} \quad \text{(A14)}$$

$$c_2 + a_2A(\tau) + \frac{b_2}{2}A^2(\tau) = \frac{dA(\tau)}{d\tau} \quad \text{(A15)}$$

With boundary conditions $A(0) = C(0) = 0$

The solution for $A(\tau)$ is given directly by

$$A(\tau) = \frac{2}{b_2}\left[\frac{u_1u_2 e^{u_1\tau} - u_1u_2 e^{u_2\tau}}{u_1 e^{u_2\tau} - u_2 e^{u_1\tau}}\right], \quad \text{(A16)}$$

Where:

$$u_1 = a_2 + \sqrt{a_2^2 - 2b_2c_2}$$
$$u_2 = a_2 - \sqrt{a_2^2 - 2b_2c_2}$$

The solution for $C(\tau)$ in (A14) is given by:

$$C(\tau) = \int_0^\tau \left(c_1 + a_1 A(\tau) + \frac{1}{2} b_1 A^2(\tau) - \frac{dB(\tau)}{d\tau} \right) du \tag{A17}$$

Where u here is the integral dummy. From (A13), $A^2(\tau) = \frac{2}{b_2}\frac{dA(\tau)}{d\tau} - \frac{2a_2}{b_2}A(\tau) - \frac{2c_2}{b_2}$, Accordingly, we can rearrange the terms inside the integral and (A17) can be written as

$$C(\tau) = \int_0^\tau \left[\frac{b_1}{b_2}\frac{dA(\tau)}{du} + \left(a_1 - \frac{b_1 a_2}{b_2} \right) A(\tau) + \left(c_1 - \frac{b_1 c_2}{b_2} \right) \right] du$$

(A18)

Distributing the integral through out the expressions, we get:

$$C(\tau) = \int_0^\tau \frac{b_1}{b_2} dA(\tau) + \int_0^\tau \left(a_1 - \frac{b_1 a_2}{b_2} \right) A(\tau) du + \int_0^\tau \left(c_1 - \frac{b_1 c_2}{b_2} \right) du \tag{A19}$$

Since $A(0) = 0$, then integrating the first and the third terms:

$$C(\tau) = \frac{b_1}{b_2}\left[A(\tau)\right] + \left(a_1 - \frac{b_1 a_2}{b_2} \right) \int_0^\tau A(\tau) du + \left(c_1 - \frac{b_1 c_2}{b_2} \right) \tau \tag{A20}$$

$$\int_0^\tau A(\tau) du = \frac{2}{b_2} \ln \left[\frac{u_2 - u_1}{u_2 e^{u_1 \tau} - u_1 e^{u_2 \tau}} \right],$$

Where u_1, and u_2 as defined above.

Accordingly, the solution for the ODE (14) is given by:

$$C(\tau) = \frac{b_1}{b_2}\left[A(\tau)\right] + \left(c_1 - \frac{b_1 c_2}{b_2} \right) \tau + \left(a_1 - \frac{b_1 a_2}{b_2} \right) \frac{2}{b_2} \ln \left[\frac{u_2 - u_1}{u_2 e^{u_1 \tau} - u_1 e^{u_2 \tau}} \right] \tag{A21}$$

Table (1)
Sources of the Monthly Data Used in the Spectral GMM Estimation

The Series	Dates	Source
3-Month Treasury Constant Maturity Rate	April 1953-Dec. 1981 Jan. 1982-Dec. 2004	McCulloch (1990) Federal Reserve Board.
1-year Treasury Constant Maturity	April 1953- Dec. 2004	Federal Reserve Board.
2-year Treasury Constant Maturity	April 1953-May 1976 June 1976- Dec. 2004	McCulloch (1990) Federal Reserve Board.
3-year Treasury Constant Maturity	April 1953-Dec.2004	Federal Reserve Board.
5-year Treasury Constant Maturity	April 1953- Dec. 2004	Federal Reserve Board.
10-year Treasury Constant Maturity	April 1953- Dec. 20034	Federal Reserve Board.
20-Year Treasury Constant Maturity Rate	April 1953- Dec. 2004	Federal Reserve Board.
Cum Dividend Stock Returns	April 1953- Dec. 2001	Robert Shiller's Home Page
Cum Dividend Stock Returns	Jan. 2002-Dec. 2004	Economagic webpage

Table (2)
Estimation of the Investment Opportunity Set Parameters,
NON-AFF1V1r-J0J: non-affine one stochastic volatility- one stochastic interest rate with jump in stock returns and jump in interest rates.
Using Spectral GMM, (April 1953-Dec.2003)

$$\frac{dS_t}{S_t} = (r_t + \mu_S)dt + \sqrt{v_t}dZ_S + \pi\sigma_r\sqrt{r_t}dZ_r + J_{Su}dN_u(\lambda_u) - J_{Sd}dN_d(\lambda_d)$$

$$dv_t = \kappa_v(\theta_v - v_t)dt + \sigma_v v_t^{\delta/2}dZ_v$$

$$dr_t = \kappa_r(\theta_r - r_t)dt + \sigma_r r_t^{\psi/2}dZ_r + J_{ru}dN_u(\lambda_u) - J_{rd}dN_d(\lambda_d)$$

$$Corr(dZ_S, dZ_v) = \rho_{sv}dt$$

Parameter	Estimate	Std. Error
μ_S	0.136	0.046
λ_u	0.021	0.013
η_{Su}	0.029	0.015
λ_d	1.412	0.481
η_{Sd}	0.027	0.010
κ_v	0.528	0.252
θ_v	0.051	0.006
σ_v	0.433	0.108
δ	1.824	0.644
ρ_{sv}	-0.311	0.170
κ_r	0.0232	0.021
θ_r	0.0342	0.025
σ_r	0.024	0.009
π	0.007	0.0058
ψ	2.652	0.552
η_{ru}	0.008	0.013
η_{rd}	0.010	0.016
λ_r	0.098	0.007

Table (3)
Asset Allocation Advices Used for Calibration*

Horizon	Risk Tolerance	Cash%	Stocks%	Bonds%	Bonds/Stocks
Short	Conservative	50	20	30	1.50
	Moderate	20	40	40	1.00
	Aggressive	5	65	30	0.46
Medium	Conservative	20	40	40	1.00
	Moderate	10	50	40	0.80
	Aggressive	0	80	20	0.25
Long	Conservative	20	45	35	0.78
	Moderate	10	60	30	0.50
	Aggressive	0	100	0	0.00

* The table constructed from the four recommendations reported by Canner, Mankiw and Weil (1997). The most conservative advice is assigned to the short horizon conservative investor, and the least conservative advice is assigned to the long horizon conservative investor. On the other hand, the most aggressive advice is assigned to the long run aggressive investor, and the least aggressive advice is assigned to the short term aggressive investor. So, the short run horizon takes the most conservative or the least aggressive of all advices, and the long run horizon takes the least conservative or the most aggressive advices. The medium horizon takes the medium of the all advices.

Table (4)
Calibrated Asset Allocation, Investor Risk Parameters, Horizon and Duration Length

Panel A: Calibrated Optimal Portfolio Choice

Horizon	Risk Tolerance	Cash%	Stocks%	Bonds%	Bonds/Stocks
Short	Conservative	30.4	32.8	36.9	1.125
	Moderate	15.3	57.6	27.1	0.47
	Aggressive	7.2	72.5	20.3	0.28
Medium	Conservative	16.1	52.1	31.8	0.61
	Moderate	11.3	65.2	23.5	0.36
	Aggressive	3.1	84.9	12	0.14
Long	Conservative	11.4	53.3	35.3	0.66
	Moderate	8.8	74.1	17.1	0.23
	Aggressive	1.9	92.6	5.5	0.06

Panel B: Calibrated Investor's Risk Parameters, Horizon and Duration length

	Parameter	Estimate	Boundary
Attitude toward risk	γ_{con}	10.26	no
	γ_{mod}	6.44	no
	γ_{agg}	1.62	no
Investment Horizons	H_{short}	5.36	no
	H_{med}	11.68	no
	H_{long}	35.00	upper
Optimal Duration of Bonds	D_{short}	4.73	lower
	D_{med}	5.85	upper
	D_{longt}	5.85	lower
Optimal Duration of Bonds	m_u	1.71	upper
	m_d	0.58	lower

Previous Publications

No	Author	Title
API/WPS 9701	جميل طاهر	النفط والتنمية المستديمة في الأقطار العربية : الفرص والتحديات
API/WPS 9702	Riad Dahel	Project Financing and Risk Analysis
API/WPS 9801	Imed Limam	A SOCIO-ECONOMIC TAXONOMY OF ARAB COUNTRIES
API/WPS 9802	محمد عدنان وديع بلقاسم العباس	منظومات المعلومات لأسواق العمل لخليجية
API/WPS 9803	Adil Abdalla	The Impact of Euro-Mediterranean Partnerships on Trade Interests of the OIC Countries
API/WPS 9804	رياض دهال حسن الحاج	حول طرق الخصخصة
API/WPS 9805	Ujjayant Chakravorty Fereidun Fesharaki Shuoying Zhou	DOMESTIC DEMAMD FOR PETROLEUM PRODUCTS IN OPEC
API/WPS 9806	Imed Limam Adil Abdalla	Inter-Arab Trade and the Potential Success of AFTA
API/WPS 9901	Karima Aly Korayem	Priorities of Social Policy Measures and the Interset of Low-Income People; the Egyptian Case
API/WPS 9902	Sami Bibi	A Welfare Analysis of the Price System Reforms' Effects on Poverty in Tunisia
API/WPS 9903	Samy Ben Naceur Mohamed Goaied	The Value Creation Process in The Tunisia Stock Exchange
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